

Mathematical analysis of plasmonic resonance for 2-D photonic crystal

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Abstract

In this article, we study the plasmonic resonance in two dimensions infinite photonic crystal mounted by the double negative nanoparticles. The corresponding physical model is described by the Helmholtz equation with so called Bloch wave condition in a periodic domain. By using the quasi-periodic layer potential techniques and the spectral theorem of quasi-periodic Neumann-Poincaré operator, the quasi-static expansion of the near field in the presence of nanoparticles is derived. Furthermore, when the magnetic permeability of nanoparticles satisfy the Drude model, we give the conditions under which the plasmonic resonance occurs, and the rate of blow up of near field energy with respect to nanoparticle's bulk electron relaxation rate and filling factor are also obtained.

Keywords: plasmonic resonance; photonic crystal; quasi-periodic Green's function; quasi-periodic Neumann-Poincaré operator; quasi-periodic layer potential; Bloch wave condition; Drude model; quasi-static regime

1 Introduction

Plasmonic resonance has been applied recently in various scientific fields, such as enhancing the brightness of light, confining strong electromagnetic fields, medical therapy, invisibility cloaking, biomedical imaging [1–11, 19] and so on.

In [5], Ammari et al. give a necessary and sufficient condition for electromagnetic power dissipation to blow up as the loss parameter of the plasmonic material goes to zero. Moreover, under some additional conditions, the cloaking due to anomalous

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localized resonance (CALR) will happen. The confocal ellipses case for plasmonic resonance and CALR are studied by Chung et al. in [4]. Ando and Kang in [12] investigate the plasmonic resonance for conductivity equation on smooth domain. The plasmonic resonance analysis for Helmholtz equation with finite frequencies is discussed by Ando, Kang and Liu in [13]. Ammari et al. in [14] give a mathematical analysis of plasmonic resonance for Helmholtz equation and apply it in the scattering and absorption enhancements, super-resolution and super-focusing. Afterward, they extend the relative results to full Maxwell equations in [15]. Recently, in [16] and [17], Ammari et al. consider the scattering by of periodic plasmonic nanoparticles mounted on a perfectly conducting sheet.

In this paper, applying the method given by [14] and [13], we give the analysis of plasmonic resonance in two dimensions infinite photonic crystal mounted by the double negative nanoparticles. Different from the model in [14], The corresponding physical model here is described by the Helmholtz equation with the Bloch wave condition (see [18]) in a periodic domain, and we only consider the wave propagation behavior inside the photonic crystal structure. It is well known that the plasmonic surface wave (Bloch wave) propagate along the interface between nanoparticles and background media, and the field intensity falls off evanescently perpendicular to the interface [24]. By using the quasi-periodic layer potential techniques and the spectral theorem of quasi-periodic Neumann-Poincaré operator, we derive the quasi-static expansion formula for the near field of nanoparticles. Moreover, in the quasi-static regime (i.e. frequency ω is small enough), the conditions closely related to the eigenvalue of quasi-periodic Neumann-Poincaré operator are obtained, under which the quasi-periodic plasmonic resonance happens. Then based on the Drude model that the magnetic permeability of nanoparticles satisfy, we get the rate of blow up of near field energy with respect to nanoparticle's bulk electron relaxation rate and filling factor.

The paper is organized as follows. In section 2, we introduce the problem formulation for plasmonic resonance in two dimensions infinite photonic crystal by using the quasi-periodic layer potential. In section 3 we derive the asymptotic expansion formula of the near field in quasi-static regime. Based on Drude model, the rate of blow up of near field energy with respect to nanoparticle's bulk electron relaxation rate and filling factor are given in section 4. Finally, we give a conclusion in section 5.

2 Problem formulation

The photonic crystal we consider in this paper consists of a homogeneous background medium which is perforated by an array of arbitrary-shaped plasmonic nanoparticles periodically along each of the two orthogonal coordinate axes in \mathbb{R}^2 .

We assume that the structure has unit periodicity and define the unit cell $Y = [0, 1]^2$. In the unit cell Y , the nanoparticle occupying a bounded and simply connected domain $D \subset Y$ whose boundary ∂D is $\mathcal{C}^{1,\alpha}$ for some $\alpha \in (0, 1)$ which is characterized by electric permittivity ε_c and magnetic permeability μ_c , while the homogeneous medium $Y \setminus \overline{D}$ is characterized by electric permittivity ε_m and magnetic permeability μ_m . In general, the propagation of light in the photonic crystal is described by the Maxwell equations. Here, we only focus on the transverse magnetic (TM) case, whose governing equations are reduced to Helmholtz equations. Furthermore, assume that the nanoparticles is dispersive, i.e. ε_c and μ_c are may depend on the frequency ω . Let $\Re \varepsilon_c < 0$, $\Im \varepsilon_c > 0$, $\Re \mu_c < 0$, $\Im \mu_c > 0$ and define

$$k_c = \omega \sqrt{\varepsilon_c \mu_c}, \quad k_m = \omega \sqrt{\varepsilon_m \mu_m}, \quad (2.1)$$

and

$$\varepsilon_D = \varepsilon_c \chi(D) + \varepsilon_m \chi(Y \setminus \overline{D}), \quad \mu_D = \mu_c \chi(D) + \mu_m \chi(Y \setminus \overline{D}), \quad (2.2)$$

where χ denotes the characteristic function. Notice that, in nano-metal materials, the electric permittivity and magnetic permeability satisfy $\Re \varepsilon_c < 0$, $\Re \mu_c < 0$ is called double negative material, which shows several unusual properties, such as, counter directance between group velocity and phase vector, negative index of refraction, reverse Doppler and Cherenkov effects [20–22]. The positive imaginary parts of electric permittivity and magnetic permeability denotes the dissipation of plasmonic nanoparticles. Moreover, we assume that ε_m, μ_m are real and strictly positive.

The propagation of wave in the photonic crystal with dipole source can be modeled by the following Helmholtz equations in infinite periodic structure [25]

$$\begin{cases} \nabla \cdot \frac{1}{\mu_D} \nabla u + \omega^2 \varepsilon_D u = a \cdot \nabla \delta_z, & \text{in } Y \setminus \partial D, \\ u|_+ = u|_-, & \text{on } \partial D, \\ \frac{1}{\mu_m} \frac{\partial u}{\partial \nu} \Big|_+ = \frac{1}{\mu_c} \frac{\partial u}{\partial \nu} \Big|_-, & \text{on } \partial D, \\ e^{-i\alpha \cdot x} u \text{ is 1-periodic in } \mathbb{R}^2 & \text{(Bloch wave condition),} \end{cases} \quad (2.3)$$

where $a \in \mathbb{R}^2$ is a constant vector and δ_z is the Dirac function at $z \in Y \setminus \overline{D}$, and $a \cdot \nabla \delta_z$ is a dipole source (see [13, 23]). The quasi-momentum $\alpha \in B = (0, 2\pi)^2$, B is called Brillouin zone [24]. In particular, when $\alpha = 0$, it is correspond to the periodic case. The last periodic condition in (2.3) is called Bloch wave condition, which characterizes the propagation behaviour of wave in the photonic crystal.

Let $G^{\alpha,k}(x, y)$ be the two-dimensional quasi-periodic Green's function for the Helmholtz equation, which is given by [25]

$$G^{\alpha,k}(x, y) = -\frac{i}{4} \sum_{n \in \mathbb{Z}^2} H_0^{(1)}(k|x - n - y|) e^{in \cdot \alpha}, \quad (2.4)$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order 0.

The quasi-periodic single layer potential of $\phi \in H^{-\frac{1}{2}}(\partial D)$ for the Helmholtz equation is defined by [25]

$$\mathcal{S}^{\alpha,k}[\phi](x) = \int_{\partial D} G^{\alpha,k}(x,y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^2, \quad (2.5)$$

and the following jump formula holds

$$\left. \frac{\partial(\mathcal{S}^{\alpha,k}[\phi])}{\partial\nu} \right|_{\pm}(x) = \left(\pm \frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha,k})^* \right) [\phi](x), \quad a.e. \ x \in \partial D, \quad (2.6)$$

where

$$(\mathcal{K}^{-\alpha,k})^*[\phi](x) = \int_{\partial D} \frac{\partial G^{\alpha,k}(x,y)}{\partial\nu(x)} \phi(y)d\sigma(y). \quad (2.7)$$

By using the quasi-periodic single layer potential (2.6) and jump formula (2.7), we can get the solution representation formula of (2.3)

Theorem 2.1 *Assume that k_c^2 is not an eigenvalue of $-\Delta$ in $Y \setminus \overline{D}$ with Dirichlet boundary condition on ∂D and the α -quasi-periodic condition on ∂Y and k_m^2 is not an eigenvalue of $-\Delta$ in D with Dirichlet boundary condition. Then the solution u to (2.3) can be represented as*

$$u(x) = \begin{cases} \mathcal{S}^{\alpha,k_c}[\phi](x), & x \in D, \\ F_z(x) + \mathcal{S}^{\alpha,k_m}[\psi](x), & x \in Y \setminus \overline{D}, \end{cases} \quad (2.8)$$

where

$$F_z(x) = \int_{\mathbb{R}^2} G^{\alpha,k_m}(x,y) a \cdot \nabla_y \delta_z(y) dy = a \cdot \nabla_x G^{\alpha,k_m}(x,z), \quad (2.9)$$

and $(\phi, \psi) \in H^{-\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ satisfy the following integral system

$$\begin{cases} \mathcal{S}^{\alpha,k_c}[\phi] - \mathcal{S}^{\alpha,k_m}[\psi] = 0, & \text{on } \partial D, \\ \frac{1}{\mu_c} \left(-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha,k_c})^* \right) [\phi] - \frac{1}{\mu_m} \left(\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha,k_m})^* \right) [\psi] = \frac{1}{\mu_m} \frac{\partial F_z}{\partial\nu}, & \text{on } \partial D. \end{cases} \quad (2.10)$$

Moreover, the mapping $u \rightarrow (\phi, \psi)$ from solutions of (2.3) to solutions of (2.10) is one-to-one.

The proof is very similar to the one in [25] (Section 8.6 Theorem 8.12).

We focus on the near field behavior in the quasi-static regime, i.e., the frequency $\omega \ll 1$. From Lemma 3.3, Lemma 3.4 and Remark 3.5 in Section 3, it finds that, for ω small enough, $\mathcal{S}^{\alpha,k}$ is invertible. Therefore, the conditions of Theorem 2.1 are

satisfied and the first equation in (2.10) becomes as

$$\phi = (\mathcal{S}^{\alpha, k_c})^{-1} \mathcal{S}^{\alpha, k_m}[\psi]. \quad (2.11)$$

Then, from the second equation in (2.10), we have that

$$\mathcal{A}(\omega)[\psi] = f, \quad (2.12)$$

where

$$\mathcal{A}(\omega) = \frac{1}{\mu_m} \left(\frac{1}{2} \mathcal{I} + (\mathcal{K}^{-\alpha, k_m})^* \right) + \frac{1}{\mu_c} \left(\frac{1}{2} \mathcal{I} - (\mathcal{K}^{-\alpha, k_c})^* \right) (\mathcal{S}^{\alpha, k_c})^{-1} \mathcal{S}^{\alpha, k_m}, \quad (2.13)$$

$$f = - \frac{1}{\mu_m} \frac{\partial F_z}{\partial \nu}. \quad (2.14)$$

Clearly,

$$\begin{aligned} \mathcal{A}(0) = \mathcal{A}_0 &= \frac{1}{\mu_m} \left(\frac{1}{2} \mathcal{I} + (\mathcal{K}^{-\alpha, 0})^* \right) + \frac{1}{\mu_c} \left(\frac{1}{2} \mathcal{I} - (\mathcal{K}^{-\alpha, 0})^* \right) \\ &= \frac{1}{2} \left(\frac{1}{\mu_m} + \frac{1}{\mu_c} \right) \mathcal{I} + \left(\frac{1}{\mu_m} - \frac{1}{\mu_c} \right) (\mathcal{K}^{-\alpha, 0})^*, \end{aligned} \quad (2.15)$$

where $(\mathcal{K}^{-\alpha, 0})^*$ is called α -quasi-periodic Neumann-Poincaré operator.

3 Asymptotic expansion of the near field

In order to solve the operator equation (2.12), we first give some basic facts about the quasi-periodic Neumann-Poincaré operator $(\mathcal{K}^{-\alpha, 0})^*$.

Lemma 3.1 (*Calderón identity*): $\mathcal{K}^{-\alpha, 0} \mathcal{S}^{\alpha, 0} = \mathcal{S}^{\alpha, 0} (\mathcal{K}^{-\alpha, 0})^*$.

The proof is very similar to the classical one in [28] and so is omitted.

Recall that $\mathcal{S}^{\alpha, 0} : H^{-\frac{1}{2}}(\partial D) \rightarrow H^{\frac{1}{2}}(\partial D)$ is not invertible in \mathbb{R}^2 . Similar to [14], we introduce a substitute of $\mathcal{S}^{\alpha, 0}$ as follows

$$\tilde{\mathcal{S}}^{\alpha, 0}[\psi] = \begin{cases} \mathcal{S}^{\alpha, 0}[\phi], & \text{if } (\psi, \chi(\partial D))_{-\frac{1}{2}, \frac{1}{2}} = 0, \\ \chi(\partial D), & \text{if } \psi = \varphi_0, \end{cases} \quad (3.1)$$

where $(\cdot, \cdot)_{-\frac{1}{2}, \frac{1}{2}}$ is the duality pairing between $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$. φ_0 is the unique eigenfunction of $(\mathcal{K}^{-\alpha, 0})^*$ associated with eigenvalue $\frac{1}{2}$ such that $(\varphi_0, \chi(\partial D))_{-\frac{1}{2}, \frac{1}{2}} = 1$.

From Lemma 3.1, we find that if $(\psi, \chi(\partial D))_{-\frac{1}{2}, \frac{1}{2}} = 0$, then $\mathcal{K}^{-\alpha, 0} \tilde{\mathcal{S}}^{\alpha, 0}[\psi] = \tilde{\mathcal{S}}^{\alpha, 0}(\mathcal{K}^{-\alpha, 0})^*[\psi]$; If $(\psi, \chi(\partial D))_{-\frac{1}{2}, \frac{1}{2}} = 1$, we have

$$\tilde{\mathcal{S}}^{\alpha, 0}(\mathcal{K}^{-\alpha, 0})^*[\psi] = \tilde{\mathcal{S}}^{\alpha, 0}(\mathcal{K}^{-\alpha, 0})^*[\varphi_0] = \frac{1}{2} \tilde{\mathcal{S}}^{\alpha, 0}[\varphi_0] = \frac{1}{2},$$

and

$$\mathcal{K}^{-\alpha, 0} \left(\tilde{\mathcal{S}}^{\alpha, 0}[\psi] \right) = \mathcal{K}^{-\alpha, 0}[\chi(\partial D)] = \frac{1}{2}.$$

Thus, we obtain the Calderón identity for $\tilde{\mathcal{S}}^{\alpha, 0}$:

$$\mathcal{K}^{-\alpha, 0} \tilde{\mathcal{S}}^{\alpha, 0} = \tilde{\mathcal{S}}^{\alpha, 0}(\mathcal{K}^{-\alpha, 0})^*. \quad (3.2)$$

In view of (3.2), we define

$$(u, v)_{\mathcal{H}^*(\partial D)} = -(u, \tilde{\mathcal{S}}^{\alpha, 0}[v])_{-\frac{1}{2}, \frac{1}{2}}. \quad (3.3)$$

Thanks to the invertibility and positivity of $-\tilde{\mathcal{S}}^{\alpha, 0}$, the inner product (3.3) leads to the self-adjointness of $(\mathcal{K}^{-\alpha, 0})^*$ and equivalence between $\mathcal{H}^*(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$.

Lemma 3.2 *Let D be a bounded simply connected domain in \mathbb{R}^2 . Then*

- (1) $(\mathcal{K}^{-\alpha, 0})^*$ is a compact self-adjoint operator in the Hilbert space $\mathcal{H}^*(\partial D)$ equipped with the inner product (3.3), which is equivalent to the original one;
- (2) Let $\{\lambda_j, \varphi_j\}$, $j = 0, 1, 2, \dots$, be the eigenvalue and normalized eigenfunction pair of $(\mathcal{K}^{-\alpha, 0})^*$, here $\lambda_0 = \frac{1}{2}$. Then, $\lambda_j \in (-\frac{1}{2}, \frac{1}{2}]$, and $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$;
- (3) $\mathcal{H}^*(\partial D) = \mathcal{H}_0^*(\partial D) \oplus \{c\varphi_0\}$, $c \in \mathbb{C}$, where $\mathcal{H}_0^*(\partial D) = \{\phi \in \mathcal{H}^*(\partial D) : \int_{\partial D} \phi d\sigma = 0\}$;
- (4) For any $\psi \in H^{-\frac{1}{2}}(\partial D)$, it hold:

$$(\mathcal{K}^{-\alpha, 0})^*[\psi] = \sum_{j=0}^{\infty} \lambda_j (\psi, \varphi_j)_{\mathcal{H}^*(\partial D)} \varphi_j. \quad (3.4)$$

The proof of Lemma 3.2 is similar to the classical case (see Lemma C.1 in [14]), we omit it here.

Notice that $(\tilde{\mathcal{S}}^{\alpha, 0})^{-1}[\chi(\partial D)] = \varphi_0$, and $-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha, 0})^* = (-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha, 0})^*) \mathcal{P}_{\mathcal{H}_0^*(\partial D)}$, where $\mathcal{P}_{\mathcal{H}_0^*(\partial D)}$ is the orthogonal projection onto $\mathcal{H}_0^*(\partial D)$. Furthermore, it follows that

$$\left(-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha, 0})^* \right) (\tilde{\mathcal{S}}^{\alpha, 0})^{-1}[\chi(\partial D)] = 0. \quad (3.5)$$

Owing to (2.15), we have

$$\mathcal{A}_0[\psi] = \sum_{j=0}^{\infty} \tau_j(\psi, \varphi_j)_{\mathcal{H}^*(\partial D)} \varphi_j, \quad (3.6)$$

where

$$\tau_j = \frac{1}{2} \left(\frac{1}{\mu_m} + \frac{1}{\mu_c} \right) + \left(\frac{1}{\mu_m} - \frac{1}{\mu_c} \right) \lambda_j. \quad (3.7)$$

From the spatial representation formula of quasi-periodic Green's function, we find

$$\begin{aligned} G^{\alpha,k}(x, y) &= -\frac{i}{4} \sum_{n \in \mathbb{Z}^2} H_0^{(1)}(k|x - n - y|) e^{in \cdot \alpha} \\ &= \sum_{n \in \mathbb{Z}^2} \frac{1}{2\pi} \ln |x - n - y| e^{in \cdot \alpha} + \tau_k \sum_{n \in \mathbb{Z}^2} e^{in \cdot \alpha} + \sum_{j=1}^{\infty} (k^{2j} \ln k) b_j \sum_{n \in \mathbb{Z}^2} |x - n - y|^{2j} e^{in \cdot \alpha} \\ &\quad + \sum_{j=1}^{\infty} k^{2j} \sum_{n \in \mathbb{Z}^2} |x - n - y|^{2j} (b_j \ln |x - n - y| + c_j) e^{in \cdot \alpha}, \end{aligned}$$

where $\tau_k = \frac{1}{2\pi}(\ln k + \gamma - \ln 2) - \frac{i}{4}$, $b_j = \frac{(-1)^j}{\pi^{2j+1}(j!)^2}$, $c_j = -b_j(\gamma - \ln 2 - \frac{i\pi}{2} - \sum_{n=1}^j \frac{1}{n})$, and γ is the Euler constant. Thus, we get the following asymptotic expansion formula of quasi-periodic single layer potential

$$\mathcal{S}^{\alpha,k} = \widehat{\mathcal{S}}^{\alpha,k} + \sum_{j=1}^{\infty} (k^{2j} \ln k) \mathcal{S}_j^{\alpha,(1)} + \sum_{j=1}^{\infty} k^{2j} \mathcal{S}_j^{\alpha,(2)}, \quad (3.8)$$

where

$$\widehat{\mathcal{S}}^{\alpha,k}[\psi](x) = \mathcal{S}^{\alpha,0}[\psi](x) + \tau_k \sum_{n \in \mathbb{Z}^2} e^{in \cdot \alpha} \int_{\partial D} \psi(y) d\sigma(y); \quad (3.9)$$

$$\mathcal{S}_j^{\alpha,(1)}[\psi](x) = \sum_{n \in \mathbb{Z}^2} \int_{\partial D} b_j |x - n - y|^{2j} e^{in \cdot \alpha} \psi(y) d\sigma(y); \quad (3.10)$$

$$\mathcal{S}_j^{\alpha,(2)}[\psi](x) = \sum_{n \in \mathbb{Z}^2} \int_{\partial D} |x - n - y|^{2j} (b_j \ln |x - n - y| + c_j) e^{in \cdot \alpha} \psi(y) d\sigma(y). \quad (3.11)$$

Moreover, we can prove (see Lemma C.2 in [14]) that, the norms $\|\mathcal{S}_j^{\alpha,(1)}\|_{\mathcal{L}(\mathcal{H}^*(\partial D), \mathcal{H}(\partial D))}$ and $\|\mathcal{S}_j^{\alpha,(2)}\|_{\mathcal{L}(\mathcal{H}^*(\partial D), \mathcal{H}(\partial D))}$ are uniformly bounded with respect to j . Furthermore, the series in (3.8) is convergent in space $\mathcal{L}(\mathcal{H}^*(\partial D), \mathcal{H}(\partial D))$.

Lemma 3.3 *As k is small enough, the operator $\mathcal{S}^{\alpha,k} : \mathcal{H}^*(\partial D) \rightarrow \mathcal{H}(\partial D)$ is invertible.*

Proof. First, we prove that, for k small enough, $\widehat{\mathcal{S}}^{\alpha,k} : \mathcal{H}^*(\partial D) \rightarrow \mathcal{H}(\partial D)$ is invertible. Since

$$\begin{aligned} (\mathcal{S}^{\alpha,0} - \widetilde{\mathcal{S}}^{\alpha,0})[\psi] &= (\mathcal{S}^{\alpha,0} - \widetilde{\mathcal{S}}^{\alpha,0})[\mathcal{P}_{\mathcal{H}_0^*(\partial D)}[\psi] + (\psi, \varphi_0)_{\mathcal{H}^*(\partial D)}\varphi_0] \\ &= (\psi, \varphi_0)_{\mathcal{H}^*(\partial D)}(\mathcal{S}^{\alpha,0}[\varphi_0] - \widetilde{\mathcal{S}}^{\alpha,0}[\varphi_0]) \\ &= (\psi, \varphi_0)_{\mathcal{H}^*(\partial D)}(\mathcal{S}^{\alpha,0}[\varphi_0] - \chi(\partial D)), \end{aligned}$$

it follows that

$$\begin{aligned} \widehat{\mathcal{S}}^{\alpha,k}[\psi] &= \widetilde{\mathcal{S}}^{\alpha,0}[\psi] + (\psi, \varphi_0)_{\mathcal{H}^*(\partial D)}(\mathcal{S}^{\alpha,0}[\varphi_0] - \chi(\partial D)) \\ &\quad + \tau_k \sum_{n \in \mathbb{Z}^2} e^{in \cdot \alpha} \int_{\partial D} (\mathcal{P}_{\mathcal{H}_0^*(\partial D)}[\psi] + (\psi, \varphi_0)_{\mathcal{H}^*(\partial D)}\varphi_0) d\sigma(y) \\ &= \widetilde{\mathcal{S}}^{\alpha,0}[\psi] + \Upsilon^{\alpha,k}[\psi], \end{aligned} \tag{3.12}$$

where

$$\Upsilon^{\alpha,k}[\psi] = (\psi, \varphi_0)_{\mathcal{H}^*(\partial D)} \left(\mathcal{S}^{\alpha,0}[\varphi_0] - \chi(\partial D) + \tau_k \sum_{n \in \mathbb{Z}^2} e^{in \cdot \alpha} \right).$$

Notice that $\widetilde{\mathcal{S}}^{\alpha,0}$ is invertible, then we have $\widehat{\mathcal{S}}^{\alpha,k}(\widetilde{\mathcal{S}}^{\alpha,0})^{-1} = \mathcal{I} + \Upsilon^{\alpha,k}(\widetilde{\mathcal{S}}^{\alpha,0})^{-1}$. Because of the compactness of $\Upsilon^{\alpha,k}$ and by the Fredholm alternative theorem, we only need to prove the injectivity of $\mathcal{I} + \Upsilon^{\alpha,k}(\widetilde{\mathcal{S}}^{\alpha,0})^{-1}$.

In fact, suppose that $v \in H^{\frac{1}{2}}(\partial D)$ satisfies $(\mathcal{I} + \Upsilon^{\alpha,k}(\widetilde{\mathcal{S}}^{\alpha,0})^{-1})[v] = 0$. According to the definition of $\widetilde{\mathcal{S}}^{\alpha,0}$ and $\Upsilon^{\alpha,k}$, if $(\widetilde{\mathcal{S}}^{\alpha,0})^{-1}[v] \in H_0^{-\frac{1}{2}}(\partial D)$, we have $(\mathcal{I} + \Upsilon^{\alpha,k}(\widetilde{\mathcal{S}}^{\alpha,0})^{-1})[v] = v$, and then $v = 0$. If $(\widetilde{\mathcal{S}}^{\alpha,0})^{-1}[v] \in \{\mu\varphi_0, \mu \in \mathbb{C}\}$, we see

$$\begin{aligned} (\mathcal{I} + \Upsilon^{\alpha,k}(\widetilde{\mathcal{S}}^{\alpha,0})^{-1})[v] &= v + \mu \left(\mathcal{S}^{\alpha,0}[\varphi_0] - \chi(\partial D) + \tau_k \sum_{n \in \mathbb{Z}^2} e^{in \cdot \alpha} \right) \\ &= \mu \left(\mathcal{S}^{\alpha,0}[\varphi_0] + \tau_k \sum_{n \in \mathbb{Z}^2} e^{in \cdot \alpha} \right), \end{aligned}$$

since we can always find a small enough k such that $\mathcal{S}^{\alpha,0}[\varphi_0] \neq -\tau_k \sum_{n \in \mathbb{Z}^2} e^{in \cdot \alpha}$, it follows that $\mu = 0$ and then $v = 0$.

Since $\widehat{\mathcal{S}}^{\alpha,k} - \mathcal{S}^{\alpha,k}$ is a compact operator and $\widehat{\mathcal{S}}^{\alpha,k}$ is invertible for k small enough. Furthermore, it is easy to prove that $\mathcal{S}^{\alpha,k}$ is injective for k small enough. In fact, we consider $\psi \in H^{-\frac{1}{2}}(\partial D)$ such that $\mathcal{S}^{\alpha,k}[\psi] = 0$. Since $u = \mathcal{S}^{\alpha,k}[\psi]$ satisfies Helmholtz

equation $\Delta u + k^2 u = 0$ in D and $Y \setminus \overline{D}$. Therefore, if k is sufficiently small such that k^2 is neither an eigenvalue of $-\Delta$ in D with the Dirichlet boundary condition on ∂D nor in $Y \setminus \overline{D}$ with the Dirichlet boundary condition on ∂D and the α -quasi-periodic condition on ∂Y . It follows that $u = 0$ and thus, $\psi = \frac{\partial u}{\partial \nu}|_+ - \frac{\partial u}{\partial \nu}|_- = 0$, as desired. By using the Fredholm alternative theorem, we see that, as k is small enough, $\mathcal{S}^{\alpha,k}$ is invertible. \square

Moreover, under some eigenvalue assumption on wave number k , we can also deduce the invertibility of $\mathcal{S}^{\alpha,k}$.

Lemma 3.4 *Suppose that k^2 is neither an eigenvalue of $-\Delta$ in D with the Dirichlet boundary condition on ∂D nor in $Y \setminus \overline{D}$ with the Dirichlet boundary condition on ∂D and the α -quasi-periodic condition on ∂Y . Then $\mathcal{S}^{\alpha,k} : H^{-\frac{1}{2}}(\partial D) \rightarrow H^{\frac{1}{2}}(\partial D)$ is invertible. Furthermore, $\mathcal{S}^{\alpha,k}$ is an isomorphism from $H^{-\frac{1}{2}}(\partial D)$ to $H^{\frac{1}{2}}(\partial D)$.*

The proof is similar to Lemma 7.2 in [25] and Theorem 7.3 in [26], so it will be given in Appendix.

Remark 3.5 *In fact, as the proof in Lemma 3.3, if k is sufficiently small, then k obviously satisfies the eigenvalue assumption of Lemma 3.4 and the corresponding results hold. Hence, the invertibility of $\mathcal{S}^{\alpha,k}$ in Lemma 3.3 can be regarded as a direct consequence of Lemma 3.4.*

Clearly, (3.8) can be written as

$$\mathcal{S}^{\alpha,k} = \widehat{\mathcal{S}}^{\alpha,k} + \mathcal{G}^{\alpha,k}, \quad (3.13)$$

where $\mathcal{G}^{\alpha,k} = (k^2 \ln k) \mathcal{S}_1^{\alpha,(1)} + k^2 \mathcal{S}_1^{\alpha,(2)} + \mathcal{O}(k^4 \ln k)$. From Lemma 3.3, we get

$$(\mathcal{S}^{\alpha,k})^{-1} = (\mathcal{I} + (\widehat{\mathcal{S}}^{\alpha,k})^{-1} \mathcal{G}^{\alpha,k})(\widehat{\mathcal{S}}^{\alpha,k})^{-1}, \quad (3.14)$$

Noting that $\|(\widehat{\mathcal{S}}^{\alpha,k})^{-1}\|_{\mathcal{L}(\mathcal{H}(\partial D), \mathcal{H}^*(\partial D))}$ is bounded for every k . Thus, as k is small enough, we have

$$(\mathcal{S}^{\alpha,k})^{-1} = (\widehat{\mathcal{S}}^{\alpha,k})^{-1} - (\widehat{\mathcal{S}}^{\alpha,k})^{-1} \mathcal{G}^{\alpha,k} (\widehat{\mathcal{S}}^{\alpha,k})^{-1} + \mathcal{O}(k^4 \ln^2 k). \quad (3.15)$$

Moreover, setting $\Lambda^{\alpha,k} = (\widetilde{\mathcal{S}}^{\alpha,0})^{-1} \widehat{\mathcal{S}}^{\alpha,k}$, then

$$\begin{aligned} \Lambda^{\alpha,k} &= (\widetilde{\mathcal{S}}^{\alpha,0})^{-1} (\widetilde{\mathcal{S}}^{\alpha,0} + \Upsilon^{\alpha,k}) = (\widetilde{\mathcal{S}}^{\alpha,0})^{-1} (\widetilde{\mathcal{S}}^{\alpha,0} + (\cdot, \varphi_0)_{\mathcal{H}^*(\partial D)} (\mathcal{S}^{\alpha,0}[\varphi_0] - \chi(\partial D) + \tau_k)) \\ &= \mathcal{I} + (\cdot, \varphi_0)_{\mathcal{H}^*(\partial D)} (\mathcal{S}^{\alpha,0}[\varphi_0] - \chi(\partial D) + \tau_k) \varphi_0. \end{aligned}$$

And then

$$(\Lambda^{\alpha,k})^{-1} = \mathcal{I} - (\cdot, \varphi_0)_{\mathcal{H}^*(\partial D)} \frac{\mathcal{S}^{\alpha,0}[\varphi_0] - \chi(\partial D) + \tau_k}{\mathcal{S}^{\alpha,0}[\varphi_0] + \tau_k} \varphi_0.$$

Hence, we find

$$\begin{aligned} (\widehat{\mathcal{S}}^{\alpha,k})^{-1} &= (\Lambda^{\alpha,k})^{-1} (\widetilde{\mathcal{S}}^{\alpha,0})^{-1} = (\widetilde{\mathcal{S}}^{\alpha,0})^{-1} - ((\widetilde{\mathcal{S}}^{\alpha,0})^{-1}[\cdot], \varphi_0)_{\mathcal{H}^*(\partial D)} \varphi_0 \\ &\quad + \frac{((\widetilde{\mathcal{S}}^{\alpha,0})^{-1}[\cdot], \varphi_0)_{\mathcal{H}^*(\partial D)}}{\mathcal{S}^{\alpha,0}[\varphi_0] + \tau_k} \varphi_0, \end{aligned}$$

then we obtain

$$\begin{aligned} (\mathcal{S}^{\alpha,k})^{-1} &= \mathcal{L}^\alpha + \mathcal{U}^{\alpha,k} - (k^2 \ln k) \mathcal{L}^\alpha \mathcal{S}_1^{\alpha,(1)} \mathcal{L}^\alpha - k^2 (\mathcal{L}^\alpha \mathcal{S}_1^{\alpha,(2)} \mathcal{L}^\alpha \\ &\quad - \ln k (\mathcal{U}^{\alpha,k} \mathcal{S}_1^{\alpha,(1)} \mathcal{L}^\alpha + \mathcal{L}^\alpha \mathcal{S}_1^{\alpha,(1)} \mathcal{U}^{\alpha,k})) + \mathcal{O}(k^2 \ln^{-1} k), \end{aligned} \quad (3.16)$$

where $\mathcal{L}^\alpha = \mathcal{P}_{\mathcal{H}_0^*(\partial D)} (\widetilde{\mathcal{S}}^{\alpha,0})^{-1}$ and $\mathcal{U}^{\alpha,k} = \frac{((\widetilde{\mathcal{S}}^{\alpha,0})^{-1}[\cdot], \varphi_0)_{\mathcal{H}^*(\partial D)}}{\mathcal{S}^{\alpha,0}[\varphi_0] + \tau_k} \varphi_0$. In particular, $\mathcal{U}^{\alpha,k} = \mathcal{O}(\ln^{-1} k)$.

Now we consider the expansion for operator $(\mathcal{K}^{-\alpha,0})^*$:

$$(\mathcal{K}^{-\alpha,k})^* = (\mathcal{K}^{-\alpha,0})^* + \sum_{j=1}^{\infty} (k^{2j} \ln k) \mathcal{K}_j^{-\alpha,(1)} + \sum_{j=1}^{\infty} k^{2j} \mathcal{K}_j^{-\alpha,(2)}, \quad (3.17)$$

where

$$\begin{aligned} \mathcal{K}_j^{-\alpha,(1)}[\psi](x) &= \sum_{n \in \mathbb{Z}^2} \int_{\partial D} b_j \frac{\partial |x - n - y|^{2j}}{\partial \nu(x)} e^{-in \cdot \alpha} \psi(y) d\sigma(y); \\ \mathcal{K}_j^{-\alpha,(2)}[\psi](x) &= \sum_{n \in \mathbb{Z}^2} \int_{\partial D} \frac{\partial |x - n - y|^{2j} (b_j \ln |x - n - y| + c_j)}{\partial \nu(x)} e^{-in \cdot \alpha} \psi(y) d\sigma(y). \end{aligned}$$

Likewise, we can prove that the norms $\|\mathcal{K}_j^{-\alpha,(1)}\|_{\mathcal{L}(\mathcal{H}^*(\partial D), \mathcal{H}^*(\partial D))}$ and $\|\mathcal{K}_j^{-\alpha,(2)}\|_{\mathcal{L}(\mathcal{H}^*(\partial D), \mathcal{H}^*(\partial D))}$ are uniformly bounded with respect to j . Furthermore, the series in (3.16) is convergent in space $\mathcal{L}(\mathcal{H}^*(\partial D), \mathcal{H}^*(\partial D))$.

Lemma 3.6 *The operator $\mathcal{A}(\omega) : \mathcal{H}^*(\partial D) \rightarrow \mathcal{H}^*(\partial D)$ has the following expansion formula:*

$$\mathcal{A}(\omega) = \mathcal{A}_0 + \omega^2 \ln \omega \mathcal{A}_1 + \mathcal{O}(\omega^2), \quad (3.18)$$

where

$$\begin{aligned} \mathcal{A}_0 &= \frac{1}{2} \left(\frac{1}{\mu_m} + \frac{1}{\mu_c} \right) \mathcal{I} + \left(\frac{1}{\mu_m} - \frac{1}{\mu_c} \right) (\mathcal{K}^{-\alpha,0})^*; \\ \mathcal{A}_1 &= \frac{1}{\mu_c} \left(\frac{1}{2} \mathcal{I} - (\mathcal{K}^{-\alpha,0})^* \right) (\widetilde{\mathcal{S}}^{\alpha,0})^{-1} \mathcal{S}_1^{\alpha,(1)} (\varepsilon_m \mu_m \mathcal{I} - \varepsilon_c \mu_c \mathcal{P}_{\mathcal{H}_0^*(\partial D)}) \end{aligned} \quad (3.19)$$

$$+ \mathcal{K}_1^{-\alpha, (1)} (\varepsilon_m \mathcal{I} - \varepsilon_c \mathcal{P}_{\mathcal{H}_0^*(\partial D)}). \quad (3.20)$$

Proof. From (3.8), (3.12) and (3.16), we have that

$$\begin{aligned} \mathcal{S}^{\alpha, k_m} &= \tilde{\mathcal{S}}^{\alpha, 0} + \Upsilon^{\alpha, k_m} + (\omega^2 \ln \omega) \varepsilon_m \mu_m \mathcal{S}_1^{\alpha, (1)} + \mathcal{O}(\omega^2), \\ (\mathcal{S}^{\alpha, k_c})^{-1} &= \mathcal{L}^\alpha + \mathcal{U}^{\alpha, k_c} - (\omega^2 \ln \omega) \varepsilon_c \mu_c \mathcal{L}^\alpha \mathcal{S}_1^{\alpha, (1)} \mathcal{L}^\alpha + \mathcal{O}(\omega^2). \end{aligned}$$

Noting the definition of Υ^{α, k_m} in (3.12), we see $\mathcal{L}^\alpha \Upsilon^{\alpha, k_m} = 0$, and then

$$\begin{aligned} (\mathcal{S}^{\alpha, k_c})^{-1} \mathcal{S}^{\alpha, k_m} &= \mathcal{P}_{\mathcal{H}_0^*(\partial D)} + \mathcal{U}^{\alpha, k_c} \tilde{\mathcal{S}}^{\alpha, 0} + \mathcal{U}^{\alpha, k_c} \Upsilon^{\alpha, k_m} \\ &\quad - (\omega^2 \ln \omega) \mathcal{L}^\alpha \mathcal{S}_1^{\alpha, (1)} (\varepsilon_m \mu_m \mathcal{I} - \varepsilon_c \mu_c \mathcal{P}_{\mathcal{H}_0^*(\partial D)}) + \mathcal{O}(\omega^2). \end{aligned}$$

By using (3.5), it yields $(-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha, 0})^*) \mathcal{U}^{\alpha, k_c} = 0$. Since $-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha, k})^* = (-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha, 0})^*) - (k^2 \ln k) \mathcal{K}_1^{-\alpha, (1)} + \mathcal{O}(k^2)$, we obtain the result. \square

Theorem 3.7 *Let $\{\tau_j(\omega), \varphi_j(\omega)\}$, $j = 0, 1, 2, \dots$, be the eigenvalue and normalized eigenfunction pair of $\mathcal{A}(\omega)$, then, in the quasi-static regime, we have the following expansion formula*

$$\tau_j(\omega) = \tau_j + (\omega^2 \ln \omega) \tau_{j,1} + \mathcal{O}(\omega^2); \quad (3.21)$$

$$\varphi_j(\omega) = \varphi_j + (\omega^2 \ln \omega) \varphi_{j,1} + \mathcal{O}(\omega^2), \quad (3.22)$$

where τ_j and φ_j are defined by (3.7) and in Lemma 3.2 respectively,

$$\tau_{j,1} = R_{jj}, \quad (3.23)$$

$$\varphi_{j,1} = \sum_{l \neq j} \frac{R_{jl}}{\left(\frac{1}{\mu_m} - \frac{1}{\mu_c}\right) (\lambda_j - \lambda_l)} \varphi_l, \quad (3.24)$$

$$R_{j,l} = (\mathcal{A}_1[\varphi_j], \varphi_l)_{\mathcal{H}^*(\partial D)}. \quad (3.25)$$

Proof. We substitute (3.18), (3.21) and (3.22) in the characteristic equation $\mathcal{A}(\omega)[\varphi_j(\omega)] = \tau_j(\omega) \varphi_j(\omega)$, and compare the coefficients of ω , ω^2 respectively, then the results hold. \square

Similar to [14], we give the definition for index set of plasmonic resonance and some mild conditions.

Definition 3.8 *We say that $J \subseteq \mathbb{N}$ is an index set of resonance if τ_j are close to zero when $j \in J$ and are bounded from below when $j \in J^c$. More precisely, we choose a threshold number $\eta_0 > 0$ independent of ω such that $|\tau_j| \geq \eta_0 > 0$, for $j \in J^c$.*

Next, we assume that the following conditions are satisfied:

(C 1) The electric permittivity and magnetic permeability $\varepsilon_m, \varepsilon_c, \mu_m, \mu_c$ are dimensionless and are of order one, the particle D has size of order one, ω is dimensionless and is of order $o(1)$.

(C 2) Each eigenvalue λ_j for $j \in J$ is a simple eigenvalue of the operator $(\mathcal{K}^{-\alpha,0})^*$.

(C 3) Let

$$\lambda(t) = \frac{1+t}{2(1-t)}. \quad (3.26)$$

We suppose that $\lambda\left(\frac{\mu_c}{\mu_m}\right) \neq 0$, i.e. $\frac{\mu_c}{\mu_m} \neq -1$.

Notice that for $j = 0$, since $\lambda_0 = \frac{1}{2}$ and (3.7), we see $\tau_0 = \frac{1}{\mu_m}$, which is of size one by our assumption. Thus, throughout this paper, we always exclude 0 from the index set J .

We define the projection operator $P_J(\omega)$ as follows

$$P_J(\omega)[\varphi_j(\omega)] = \begin{cases} \varphi_j(\omega), & j \in J, \\ 0, & j \in J^c. \end{cases} \quad (3.27)$$

Let $\{\tilde{\tau}_j(\omega); \tilde{\varphi}_j(\omega)\}$ be the eigenvalue system of $\mathcal{A}^*(\omega)$, we can get the similar expansion formula

$$\tilde{\tau}_j(\omega) = \overline{\tau_j(\omega)}; \quad (3.28)$$

$$\tilde{\varphi}_j(\omega) = \varphi_j + (\omega^2 \ln \omega) \tilde{\varphi}_{j,1} + \mathcal{O}(\omega^2). \quad (3.29)$$

Applying the eigenfunction expansion, we have

$$P_J(\omega)[x] = \sum_{j \in J} (x, \tilde{\varphi}_j(\omega))_{\mathcal{H}^*(\partial D)} \varphi_j(\omega), \quad x \in \mathcal{H}^*(\partial D). \quad (3.30)$$

Hence, from (2.12), we obtain

$$\psi = \mathcal{A}^{-1}(\omega)[f] = \sum_{j \in J} \frac{(f, \tilde{\varphi}_j(\omega))_{\mathcal{H}^*(\partial D)}}{\tau_j(\omega)} \varphi_j(\omega) + \mathcal{A}^{-1}(\omega)[P_{J^c}(\omega)[f]], \quad (3.31)$$

and it can be proved that, as ω is small enough, $\|\mathcal{A}^{-1}(\omega)P_{J^c}(\omega)\| \lesssim 1 + \mathcal{O}(\omega)$ (see Lemma 2.5 in [14]). Here and throughout this paper, $A \lesssim B$ means $A \leq CB$ for some positive constant C independent of parameters involved. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

Theorem 3.9 *In the quasi-static regime, the near field u has the following repre-*

sentation

$$u(x) = \begin{cases} \mathcal{S}^{\alpha, k_c}[\phi](x), & x \in D, \\ F_z(x) + \mathcal{S}^{\alpha, k_m}[\psi](x), & x \in Y \setminus \overline{D}, \end{cases} \quad (3.32)$$

where F_z is defined by (2.9), $(\phi, \psi) \in H^{-\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ is given by

$$\phi = (\mathcal{S}^{\alpha, k_c})^{-1} \mathcal{S}^{\alpha, k_m}[\psi], \quad (3.33)$$

$$\psi = \sum_{j \in J} \frac{\omega (f_1, \varphi_j)_{\mathcal{H}^*(\partial D)} + \mathcal{O}(\omega^3 \ln \omega)}{\lambda \left(\frac{\mu_c}{\mu_m} \right) - \lambda_j + \mathcal{O}(\omega^3 \ln \omega)} \varphi_j + \mathcal{O}(\omega) \quad (3.34)$$

with $\lambda(t)$ being defined by (3.26), and

$$f_1 = \frac{i\mu_m\mu_c\sqrt{\varepsilon_m\mu_m}}{4(\mu_c - \mu_m)} \left(\sum_{n \in \mathbb{Z}^2} \left(aH_1^{(1)}(k_m|x - n - z|) - a \cdot (x - n - z)k_mH_2^{(1)}(k_m|x - n - z|) \frac{(x - n - z)}{|x - n - z|} \right) \frac{e^{in \cdot \alpha}}{|x - n - z|}, \nu(x) \right), \quad (3.35)$$

here $H_1^{(1)}$, $H_2^{(1)}$ denote the Hankel functions of the first of order 1 and the Hankel functions of the second of order 1 respectively.

Proof. According to the representation formula of quasi-periodic Green's function, it is easy to find that

$$f = -\frac{\partial F_z}{\partial \nu} = \omega f_1, \quad (3.36)$$

where f_1 is defined by (3.35). Then, by (3.7), (3.29), (3.31) and Theorem 3.7, we find

$$\psi = \mathcal{A}^{-1}(\omega)[f] = \sum_{j \in J} \frac{\omega (f_1, \varphi_j)_{\mathcal{H}^*(\partial D)} + \mathcal{O}(\omega^3 \ln \omega)}{\frac{1}{2} \left(\frac{1}{\mu_m} + \frac{1}{\mu_c} \right) + \left(\frac{1}{\mu_m} - \frac{1}{\mu_c} \right) \lambda_j + \mathcal{O}(\omega^3 \ln \omega)} \varphi_j + \mathcal{O}(\omega). \quad (3.37)$$

Thus, from (2.11) and solution formula (2.8), it yields the result. \square

Remark 3.10 According to Theorem 3.9, we can see that if $\lambda \left(\frac{\mu_c}{\mu_m} \right) \rightarrow \lambda_j$ or $\tau_j \rightarrow 0$, for some $j \in J$, the near field $u(x)$, $x \in D$ blows up, i.e., plasmonic resonance occurs. Furthermore, in next section, we will analysis the rate of blow up of near field energy.

4 Analysis for the rate of blow up of near field energy

In order to estimate the rate of blow up in near field energy, we first give the estimation for gradient of the solution.

Lemma 4.1 *Let $\phi = \tilde{\phi} + (\phi, \varphi_0)\varphi_0$, here $\tilde{\phi} \in \mathcal{H}_0^*(\partial D)$, for the solution of (2.3) in D , i.e., $u = \mathcal{S}^{\alpha, k_c}[\phi]$, we have the following estimation*

$$\left| \|\nabla u\|_{L^2(D)}^2 - \|\tilde{\phi}\|_{\mathcal{H}^*(\partial D)}^2 \right| \lesssim |\omega \ln \omega|^2 |(\phi, \varphi_0)|^2. \quad (4.1)$$

Proof. Using divergence theorem in D , we have that

$$\int_D |\nabla u|^2 dx \leq |k_c|^2 \int_D |u|^2 dx + \left| \int_{\partial D} u \overline{\frac{\partial u}{\partial \nu}} \Big|_- d\sigma \right|.$$

By (3.8) and solution formula (2.8), it deduces

$$|u|^2 = |\mathcal{S}^{\alpha, k_c}[\phi]|^2 \lesssim \left| \widehat{\mathcal{S}}^{\alpha, k_c}[\phi] \right|^2 + \left| \sum_{j=1}^{\infty} (k_c^{2j} \ln k_c) \mathcal{S}_j^{\alpha, (1)}[\phi] \right|^2 + \left| \sum_{j=1}^{\infty} k_c^{2j} \mathcal{S}_j^{\alpha, (2)}[\phi] \right|^2.$$

Since $\tau_{k_c} \lesssim \ln k_c$ and ω is small enough, we get

$$\|u\|_{L^2(D)}^2 \lesssim |\ln k_c| \|\phi\|_{\mathcal{H}^*(\partial D)}^2. \quad (4.2)$$

Noting the Poisson summation formula

$$\sum_{n \in \mathbb{Z}^2} e^{in \cdot \alpha} = \sum_{n \in \mathbb{Z}^2} \delta\left(\frac{\alpha}{2\pi} - n\right),$$

from (3.8), (3.17) and the jump formula, for every $\alpha \in (0, 2\pi)^2$, we get

$$\begin{aligned} \left| \int_{\partial D} u \overline{\frac{\partial u}{\partial \nu}} \Big|_- d\sigma \right| &= \left| \int_{\partial D} \mathcal{S}^{\alpha, k_c}[\phi] \overline{\frac{\partial}{\partial \nu} \mathcal{S}^{\alpha, k_c}[\phi]} \Big|_- d\sigma \right| \\ &= \left| \int_{\partial D} \mathcal{S}^{\alpha, k_c}[\phi] \overline{\left(-\frac{1}{2} \mathcal{I} + (\mathcal{K}^{-\alpha, k_c})^* \right) [\phi]} d\sigma \right| \\ &\leq \left| \int_{\partial D} \mathcal{S}^{\alpha, 0}[\phi] \overline{\left(-\frac{1}{2} \mathcal{I} + (\mathcal{K}^{-\alpha, 0})^* \right) [\phi]} d\sigma \right| + |E|, \end{aligned}$$

where

$$\begin{aligned}
 E &= \int_{\partial D} \mathcal{S}^{\alpha,0}[\phi] \overline{\sum_{j=1}^{\infty} (k_c^{2j} \ln k_c) \mathcal{K}_j^{-\alpha,(1)}[\phi] d\sigma} + \int_{\partial D} \mathcal{S}^{\alpha,0}[\phi] \overline{\sum_{j=1}^{\infty} k_c^{2j} \mathcal{K}_j^{-\alpha,(2)}[\phi] d\sigma} \\
 &+ \int_{\partial D} \sum_{j=1}^{\infty} (k_c^{2j} \ln k_c) \mathcal{S}_j^{\alpha,(1)}[\phi] \overline{\left(-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha,k_c})^*\right) [\phi] d\sigma} \\
 &+ \int_{\partial D} \sum_{j=1}^{\infty} k_c^{2j} \mathcal{S}_j^{\alpha,(2)}[\phi] \overline{\left(-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha,k_c})^*\right) [\phi] d\sigma}.
 \end{aligned}$$

For k_c small enough, it is easy to find that $|E| \lesssim |k_c \ln k_c|^2 \|\phi\|_{\mathcal{H}^*(\partial D)}^2$. Next, we estimate $\left| \int_{\partial D} \mathcal{S}^{\alpha,0}[\phi] \overline{\left(-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha,0})^*\right) [\phi] d\sigma} \right|$.

Since $(\mathcal{K}^{-\alpha,k_c})^*[\varphi_0] = \frac{1}{2}\varphi_0$ and $\mathcal{S}^{\alpha,0}[\varphi_0] = 0$, for $\phi = \tilde{\phi} + (\phi, \varphi_0)\varphi_0$, we see that

$$\begin{aligned}
 &\int_{\partial D} \mathcal{S}^{\alpha,0}[\phi] \overline{\left(-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha,0})^*\right) [\phi] d\sigma} = \int_{\partial D} \mathcal{S}^{\alpha,0}[\tilde{\phi}] \overline{\left(-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha,0})^*\right) [\tilde{\phi}] d\sigma} \\
 &= \int_{\partial D} \mathcal{S}^{\alpha,0} \left[\sum_{j=1}^{\infty} (\tilde{\phi}, \varphi_j) \varphi_j \right] \overline{\left(-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha,0})^*\right) \left[\sum_{l=1}^{\infty} (\tilde{\phi}, \varphi_l) \varphi_l \right] d\sigma} \\
 &= \int_{\partial D} \sum_{j=1}^{\infty} (\tilde{\phi}, \varphi_j) \mathcal{S}^{\alpha,0}[\varphi_j] \overline{\sum_{l=1}^{\infty} (\tilde{\phi}, \varphi_l) \left(-\frac{1}{2} + \lambda_l\right) \varphi_l d\sigma} \\
 &= \sum_{j,l=1}^{\infty} \left(-\frac{1}{2} + \lambda_l\right) \overline{(\tilde{\phi}, \varphi_l)(\tilde{\phi}, \varphi_j)} \int_{\partial D} \mathcal{S}^{\alpha,0}[\varphi_j] \overline{\varphi_l} d\sigma,
 \end{aligned}$$

Notice that $\int_{\partial D} \mathcal{S}^{\alpha,0}[\varphi_j] \overline{\varphi_l} d\sigma = -(\varphi_l, \varphi_j)_{\mathcal{H}^*(\partial D)} = -\delta_{lj}$ (the Kronecker's delta), we have that

$$\int_{\partial D} \mathcal{S}^{\alpha,0}[\phi] \overline{\left(-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha,0})^*\right) [\phi] d\sigma} = \sum_{j=1}^{\infty} \left(-\frac{1}{2} + \lambda_j\right) \left|(\tilde{\phi}, \varphi_j)\right|^2.$$

Since $\lambda_j \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ ($j \geq 1$), it implies

$$\left| \int_{\partial D} \mathcal{S}^{\alpha,0}[\phi] \overline{\left(-\frac{1}{2}\mathcal{I} + (\mathcal{K}^{-\alpha,0})^*\right) [\phi] d\sigma} \right| \approx \|\tilde{\phi}\|_{\mathcal{H}^*(\partial D)}^2. \quad (4.3)$$

Combing (4.2), (4.3) and the estimation of E , we obtain the result. \square

For simplicity, let $\sigma = \Re\left(\frac{1}{\mu_c(\omega)}\right)$, $\delta = \Im\left(\frac{1}{\mu_c(\omega)}\right) < 0$, then τ_j given by (3.7) can be written as

$$\tau_j = \frac{1}{2}(\sigma + \mu_m^{-1}) - (\sigma - \mu_m^{-1})\lambda_j + \delta\left(\frac{1}{2} - \lambda_j\right)i. \quad (4.4)$$

Next, we define the approximate density ψ_0 as the solution of equation $\mathcal{A}_0[\psi_0] = f$, here \mathcal{A}_0 is given by (3.19), then, similar to (3.31), applying the eigenfunction expansion, it follows that

$$\psi_0 = \mathcal{A}_0^{-1}[f] = \sum_{j=0}^{\infty} \frac{(f, \varphi_j)_{\mathcal{H}^*(\partial D)}}{\tau_j} \varphi_j. \quad (4.5)$$

Lemma 4.2 *Under Conditions (C1), (C2) and (C3), ψ_0 is given by (4.5) and has the decomposition $\psi_0 = \tilde{\psi}_0 + c\varphi_0$, ($\tilde{\psi}_0 \in \mathcal{H}^*(\partial D)$, c is a constant). Moreover, let $\sigma = \Re(\mu_c^{-1}(\omega))$, $\delta = \Im(\mu_c^{-1}(\omega))$. Then, for sufficiently small $|\delta|$, we have that*

(1) $\|\mathcal{A}_0^{-1}\|_{\mathcal{L}(\mathcal{H}^*(\partial D), \mathcal{H}^*(\partial D))} \lesssim \delta^{-1}$.

(2) If $\lambda(\sigma^{-1}\mu_m^{-1}) \neq \lambda_j$, ($j \geq 0$), then $\|\mathcal{A}_0^{-1}\|_{\mathcal{L}(\mathcal{H}^*(\partial D), \mathcal{H}^*(\partial D))} \lesssim C$ for some positive constant C .

(3) If $\lambda(\sigma^{-1}\mu_m^{-1}) = \lambda_j$ for some $j \geq 1$, then $\|\tilde{\psi}_0\|_{\mathcal{H}^*(\partial D)} \gtrsim |\delta|^{-1} \left| (f, \varphi_j)_{\mathcal{H}^*(\partial D)} \right|$.

Proof. (1) Since, for $j \neq 0$, $|\tau_j^{-1}| \lesssim \frac{1}{|\delta|(\frac{1}{2} - \lambda_j)} \lesssim |\delta|^{-1}$, then it find

$$\|\psi_0\|_{\mathcal{H}^*(\partial D)}^2 \lesssim |\delta|^{-2} \sum_{j=0}^{\infty} \left| (f, \varphi_j)_{\mathcal{H}^*(\partial D)} \right|^2 \lesssim |\delta|^{-2} \|f\|_{\mathcal{H}^*(\partial D)}^2. \quad (4.6)$$

Thus, $\|\mathcal{A}_0^{-1}\|_{\mathcal{L}(\mathcal{H}^*(\partial D), \mathcal{H}^*(\partial D))} \lesssim |\delta|^{-1}$.

(2) If $\lambda(\sigma^{-1}\mu_m^{-1}) \neq \lambda_j$, then, for $j \geq 0$, we find $|\lambda(\sigma\mu_m^{-1}) - \lambda_j| \geq c_0$, here c_0 is a positive constant. Hence, $|\tau_j^{-1}| \lesssim 1$ and $\|\psi_0\|_{\mathcal{H}^*(\partial D)}^2 \lesssim \|f\|_{\mathcal{H}^*(\partial D)}^2$, i.e., $\|\mathcal{A}_0^{-1}\|_{\mathcal{L}(\mathcal{H}^*(\partial D), \mathcal{H}^*(\partial D))} \lesssim C$.

(3) If $\lambda(\sigma^{-1}\mu_m^{-1}) = \lambda_j$ for some $j \geq 1$, then

$$\|\tilde{\psi}_0\|_{\mathcal{H}^*(\partial D)} \gtrsim \left| (\tilde{\psi}_0, \varphi_j)_{\mathcal{H}^*(\partial D)} \right| \gtrsim \frac{\left| (f, \varphi_j)_{\mathcal{H}^*(\partial D)} \right|}{\left| \delta(\frac{1}{2} - \lambda_j) \right|} \gtrsim |\delta|^{-1} \left| (f, \varphi_j)_{\mathcal{H}^*(\partial D)} \right|. \quad (4.7)$$

□

Theorem 4.3 *Let u be the solution of (2.3), and $\sigma = \Re(\mu_c^{-1}(\omega))$, $\delta = \Im(\mu_c^{-1}(\omega))$. Suppose that $|\delta|^{-1}\omega \leq c_1$ for sufficiently small c_1 , we have*

(1) If $\lambda(\sigma^{-1}\mu_m^{-1}) \neq \lambda_j$ for any $j \geq 0$, then there exist a constant C independent of δ such that

$$\|\nabla u\|_{L^2(D)} \leq C. \quad (4.8)$$

(2) If $\lambda(\sigma^{-1}\mu_m^{-1}) = \lambda_j$ for some $j \geq 1$, let z be such that $a \cdot \nabla_z \mathcal{S}^{\alpha,0}[\varphi_j](z) \neq 0$. Then

$$\|\nabla u\|_{L^2(D)} \approx |\delta|^{-1} \quad (4.9)$$

as $|\delta|$ is small enough.

Proof. (1) From (3.18), it follows that

$$\mathcal{A}(\omega) = \mathcal{A}_0 \left(\mathcal{I} + \mathcal{A}_0^{-1} (\omega^2 \ln \omega \mathcal{A}_1 + \mathcal{O}(\omega^2)) \right), \quad (4.10)$$

and then

$$\psi = \left(\mathcal{I} + \mathcal{A}_0^{-1} (\omega^2 \ln \omega \mathcal{A}_1 + \mathcal{O}(\omega^2)) \right)^{-1} \mathcal{A}_0^{-1}[f]. \quad (4.11)$$

By using Lemma 4.2 (1), we see

$$\left\| \mathcal{A}_0^{-1} (\omega^2 \ln \omega \mathcal{A}_1 + \mathcal{O}(\omega^2)) \right\|_{\mathcal{L}(\mathcal{H}^*(\partial D), \mathcal{H}^*(\partial D))} \lesssim |\delta|^{-1} \omega. \quad (4.12)$$

Therefore,

$$\begin{aligned} \|\psi - \psi_0\|_{\mathcal{H}^*(\partial D)} &= \left\| \left(\mathcal{I} + \mathcal{A}_0^{-1} (\omega^2 \ln \omega \mathcal{A}_1 + \mathcal{O}(\omega^2)) \right)^{-1} \mathcal{A}_0^{-1}[f] - \mathcal{A}_0^{-1}[f] \right\|_{\mathcal{H}^*(\partial D)} \\ &\lesssim |\delta|^{-1} \omega \left\| \mathcal{A}_0^{-1}[f] \right\|_{\mathcal{H}^*(\partial D)} \\ &= |\delta|^{-1} \omega \left\| \psi_0 \right\|_{\mathcal{H}^*(\partial D)}. \end{aligned} \quad (4.13)$$

If $\lambda(\sigma^{-1}\mu_m^{-1}) \neq \lambda_j$, by Lemma 4.2 (2), we get

$$\|\psi\|_{\mathcal{H}^*(\partial D)} \lesssim (1 + |\delta|^{-1} \omega) \|\psi_0\|_{\mathcal{H}^*(\partial D)} = (1 + |\delta|^{-1} \omega) \left\| \mathcal{A}_0^{-1}[f] \right\|_{\mathcal{H}^*(\partial D)} \lesssim \|f\|_{\mathcal{H}^*(\partial D)}.$$

Then, by Lemma 4.1, Lemma 3.4 and Remark 3.5, it yields

$$\|\nabla u\|_{L^2(D)} = \left\| \nabla \mathcal{S}^{\alpha, k_c}[\phi] \right\|_{\mathcal{H}(\partial D)} \lesssim \|\phi\|_{\mathcal{H}^*(\partial D)} \lesssim \|\psi\|_{\mathcal{H}^*(\partial D)} \lesssim \|f\|_{\mathcal{H}^*(\partial D)}.$$

(2) If $\lambda(\sigma^{-1}\mu_m^{-1}) = \lambda_j$, ($j \geq 1$), then, from Lemma 4.2 (3), we find

$$\|\tilde{\psi}_0\|_{\mathcal{H}^*(\partial D)} \gtrsim |\delta|^{-1} \left| (f, \varphi_j)_{\mathcal{H}^*(\partial D)} \right|. \quad (4.14)$$

Moreover, by (4.13), it deduces

$$|\delta|^{-1}\omega \|\psi_0\|_{\mathcal{H}^*(\partial D)} \gtrsim \|\psi - \psi_0\|_{\mathcal{H}^*(\partial D)} \gtrsim \|\tilde{\psi}_0\|_{\mathcal{H}^*(\partial D)} - \|\tilde{\psi}\|_{\mathcal{H}^*(\partial D)}, \quad (4.15)$$

where $\tilde{\psi} = \psi - (\psi, \varphi_0)_{\mathcal{H}^*(\partial D)}\varphi_0$.

Combining now (4.14) and (4.15), and noting $|\delta|^{-1}\omega \leq c_1$, for sufficiently small $|\delta|$, we have that

$$\|\tilde{\psi}\|_{\mathcal{H}^*(\partial D)} \gtrsim \|\tilde{\psi}_0\|_{\mathcal{H}^*(\partial D)} - |\delta|^{-1}\omega \|\psi_0\|_{\mathcal{H}^*(\partial D)} \gtrsim |\delta|^{-1} \left| (f, \varphi_j)_{\mathcal{H}^*(\partial D)} \right|. \quad (4.16)$$

Hence, we obtain from Lemma 4.1 and Remark 3.5 that

$$\begin{aligned} \|\nabla u\|_{L^2(D)}^2 &= \|\nabla \mathcal{S}^{\alpha, k_c}[\phi]\|_{\mathcal{H}(\partial D)}^2 \gtrsim \|\tilde{\phi}\|_{\mathcal{H}^*(\partial D)}^2 - |\omega \ln \omega|^2 |(\phi, \varphi_0)_{\mathcal{H}^*(\partial D)}|^2 \\ &\gtrsim \|\tilde{\psi}\|_{\mathcal{H}^*(\partial D)}^2 - |\omega \ln \omega|^2 |(\psi, \varphi_0)_{\mathcal{H}^*(\partial D)}|^2 \\ &\gtrsim |\delta|^{-2} |(f, \varphi_j)_{\mathcal{H}^*(\partial D)}|^2 - |\omega \ln \omega|^2 |(\psi, \varphi_0)_{\mathcal{H}^*(\partial D)}|^2. \end{aligned}$$

Next, we will show that $|(\psi, \varphi_0)_{\mathcal{H}^*(\partial D)}|$ is bounded, and $(f, \varphi_j)_{\mathcal{H}^*(\partial D)} \neq 0$.

Let $(\mathcal{I} + (\omega^2 \ln \omega \mathcal{A}_1 + \mathcal{O}(\omega^2)) \mathcal{A}_0^{-1})^{-1}[f] = g$. Since $|\delta|^{-1}\omega \leq c_1$, by (4.12), it implies $\|g\|_{\mathcal{H}^*(\partial D)}$ is bounded. Then, noting $\lambda_0 = \frac{1}{2}$, we see that

$$|(\psi, \varphi_0)_{\mathcal{H}^*(\partial D)}| = |\tau_0^{-1}(g, \varphi_0)_{\mathcal{H}^*(\partial D)}| = |\mu_m(g, \varphi_0)_{\mathcal{H}^*(\partial D)}| \lesssim C.$$

Here, C is a positive constant.

Applying Green's formula and jump relation, it follows that

$$\begin{aligned} (f, \varphi_j)_{\mathcal{H}^*(\partial D)} &= - \left(\frac{\partial F_z}{\partial \nu}, \varphi_j \right)_{\mathcal{H}^*(\partial D)} = \left(\frac{\partial F_z}{\partial \nu}, \tilde{\mathcal{S}}^{\alpha, 0}[\varphi_j] \right) \\ &= \int_D \mathcal{S}^{\alpha, 0}[\varphi_j] \Delta F_z dx - \int_D F_z \Delta \mathcal{S}^{\alpha, 0}[\varphi_j] dx + \int_{\partial D} F_z \frac{\partial \mathcal{S}^{\alpha, 0}}{\partial \nu}[\varphi_j] \Big|_- d\sigma \\ &= -k_m^2 \int_D F_z \mathcal{S}^{\alpha, 0}[\varphi_j] dx + \left(\lambda_j - \frac{1}{2} \right) \int_{\partial D} F_z \varphi_j d\sigma. \end{aligned}$$

Furthermore, from (3.8) and (2.9), it yields

$$\begin{aligned} \int_{\partial D} F_z \varphi_j d\sigma &= \int_{\partial D} a \cdot \nabla_x G^{\alpha, k_m}(x, z) \varphi_j d\sigma = - \int_{\partial D} a \cdot \nabla_z G^{\alpha, k_m}(x, z) \varphi_j d\sigma \\ &= -a \cdot \nabla_z \mathcal{S}^{\alpha, k_m}[\varphi_j](z) = -a \cdot \nabla_z \mathcal{S}^{\alpha, 0}[\varphi_j](z) + \mathcal{O}(\omega^2 \ln \omega). \end{aligned}$$

Thereby, $(f, \varphi_j)_{\mathcal{H}^*(\partial D)} = -a \left(\lambda_j - \frac{1}{2} \right) \cdot \nabla_z \mathcal{S}^{\alpha,0}[\varphi_j](z) + \mathcal{O}(\omega^2 \ln \omega)$. Clearly, for ω small enough, $(f, \varphi_j)_{\mathcal{H}^*(\partial D)} \neq 0$. We find $\|\nabla u\|_{L^2(D)} \gtrsim |\delta|^{-1}$, and likewise $|\delta|^{-1} \gtrsim \|\nabla u\|_{L^2(D)}$ can also be proved. Hence, (4.17) hold. \square

Remark 4.4 *From the proof of Theorem 4.3 (2), it shows that If $\lambda(\sigma^{-1}\mu_m^{-1}) = \lambda_j$ for some $j \geq 1$, in the quasi-static regime and for fixed parameter δ , the near energy*

$$\|\nabla u\|_{L^2(D)} \gtrsim \left| \frac{a}{\delta} \left(\lambda_j - \frac{1}{2} \right) \cdot \nabla_z \mathcal{S}^{\alpha,0}[\varphi_j](z) \right|. \quad (4.17)$$

Then, because of the singularity of $\nabla_z \mathcal{S}^{\alpha,0}[\varphi_j](z)$, we find that when the the location of the dipole source $z \rightarrow \partial D$ or the intensity of the dipole source $|a|$ is large enough, the near energy will become large, which indicates the enhancement of near field.

Now we write $\mu_c(\omega) = \mu^{(1)}(\omega) + i\mu^{(2)}(\omega)$, and $\mu^{(1)}, \mu^{(2)}$ satisfy the following Kramer-Kronig relations (Hilbert transform) [6, 14]:

$$\mu^{(1)}(\omega) = \frac{1}{\pi} P.V. \int_{-\infty}^{+\infty} \frac{\mu^{(2)}(s)}{\omega - s} ds; \quad (4.18)$$

$$\mu^{(2)}(\omega) = -\frac{1}{\pi} P.V. \int_{-\infty}^{+\infty} \frac{\mu^{(1)}(s)}{\omega - s} ds. \quad (4.19)$$

The magnetic permeability of particle $\mu_c(\omega)$ can be described by the Drude model [1, 6, 14, 27], i.e.

$$\mu_c(\omega) = \mu_0 \left(1 - F \frac{\omega^2}{\omega^2 - \omega_0^2 + i\omega\tau^{-1}} \right), \quad (4.20)$$

where $\tau > 0$ is the nanoparticle's bulk electron relaxation rate (τ^{-1} is the damping coefficient), F is a filling factor, μ_0 is permeability of free space, and ω_0 is a localized plasmon resonant frequency. In particular, as $\omega \in \mathbb{R}$, if

$$(1 - F)(\omega^2 - \omega_0^2)^2 - F\omega_0^2(\omega^2 - \omega_0^2) + \tau^{-2}\omega^2 < 0, \quad (4.21)$$

we see the real part of $\mu_c(\omega)$ is negative, i.e., $\mu^{(1)}(\omega) < 0$, which indicates some singular property of particles.

Theorem 4.5 *Let u be the solution of (2.3). The magnetic permeability of particle $\mu_c(\omega) = \mu^{(1)}(\omega) + i\mu^{(2)}(\omega)$ satisfies the Kramer-Kronig relations (4.18) and the Drude model (4.20). We find that, in the quasi-static regime, if $\lambda \left(\frac{|\mu_c(\omega)|^2}{\mu_m \mu^{(1)}(\omega)} \right) = \lambda_j$ for some $j \geq 1$, and $a \cdot \nabla_z \mathcal{S}^{\alpha,0}[\varphi_j](z) \neq 0$ for some $z \in Y \setminus \overline{D}$, then, as the nanoparticle's bulk*

electron relaxation rate $\tau \rightarrow 0^+$ or the filling factor $F \rightarrow 0^+$, the near field energy

$$\|\nabla u\|_{L^2(D)} \rightarrow \infty. \quad (4.22)$$

Proof. In Theorem 4.3 (2), let $|\delta| = \frac{\mu^{(2)}(\omega)}{|\mu_c(\omega)|^2} = \frac{\tau F \mu_0 \omega^3}{(\tau^2(\omega^2 - \omega_0^2)^2 + \omega^2)|\mu_c(\omega)|^2}$, we get the result. \square

Remark 4.6 From Theorem 4.5, we see that, if the filling factor F is fixed, then the rate of blows up of near field energy is $\mathcal{O}(\tau^{-1})$, (as $\tau \rightarrow 0^+$). Likewise, fixing $\tau > 0$, the rate of blows up of near field energy is $\mathcal{O}(F^{-1})$, (as $F \rightarrow 0^+$).

Remark 4.7 As application, we can solve a simple optical design problem: How to determine the magnetic permeability of nanoparticles embedded into the optical device, such that the near field energy is maximized? In fact, according to Theorem 4.5, for fixed sufficiently small frequency ω , giving the magnetic permeability of background medium μ_m and filling factor F , then we can obtain the nanoparticle's bulk electron relaxation rate τ by the following direct algorithm:

- (i) Calculate the eigenvalue λ_j ($j \geq 1$) for α -quasi-periodic Neumann-Poincaré operator $(\mathcal{K}^{-\alpha,0})^*$;
- (ii) Using the Drude model (4.20), and find the bulk electron relaxation rate τ by solving the nonlinear equation

$$\lambda \left(\frac{|\mu_c(\tau)|^2}{\mu_m \mu^{(1)}(\tau)} \right) = \lambda_j,$$

where the function $\lambda(t)$ is defined by (3.26).

Meanwhile, by applying above algorithm, we can similarly compute the filling factor F . From the Drude model, it is easy to see that the magnetic permeability of nanoparticles are fully determined by the bulk electron relaxation rate and filling factor.

Moreover, the corresponding near field can be computed by (3.32). Notice that the solution of (2.3) and wave number k_c are complex, by using Green's formula, the resonance near field energy can be written as

$$\begin{aligned} \|\nabla u\|_{L^2(D)}^2 &= ((\Re k_c)^2 - (\Im k_c)^2) \|\mathcal{S}^{\alpha, k_c}[\phi]\|_{L^2(D)}^2 \\ &\quad + \Re \int_{\partial D} \mathcal{S}^{\alpha, k_c}[\phi] \overline{\left(-\frac{1}{2} \mathcal{I} + (\mathcal{K}^{-\alpha, k_c})^* \right) [\phi]} d\sigma, \end{aligned}$$

which can be calculated by the Nyström method (see [29]).

5 Conclusion

The mathematical analysis of plasmonic resonance of two dimensions photonic crystal in which embedded the double negative nanoparticles was given in this paper. Based on perturbation theorem, by applying the quasi-periodic layer potential method and the spectral theorem of quasi-periodic Neumann-Poincaré operator, we have deduced the quasi-static expansion of the near field and shown that when $\lambda \left(\frac{\mu_c}{\mu_m} \right)$ is close to some eigenvalue of quasi-periodic Neumann-Poincaré operator, the near field will enhance obviously. Moreover, as the magnetic permeability of nanoparticles was described by the Drude model, the conditions under which the plasmonic resonance occurs, and the estimate for rate of blow up of near field energy with respect to nanoparticle's bulk electron relaxation rate and filling factor were also obtained.

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6 Appendix

The proof of Lemma 3.4:

Proof. It is sufficient to prove the last part of the Lemma 3.4, i.e. $\mathcal{S}^{\alpha,k}$ is an isomorphism from $H^{-\frac{1}{2}}(\partial D)$ to $H^{\frac{1}{2}}(\partial D)$. We first show the sesquilinear form $(\mathcal{S}^{\alpha,i}[\cdot], \cdot)_{\frac{1}{2}, -\frac{1}{2}}$ on $H^{-\frac{1}{2}}(\partial D)$ is coercive, here, $\mathcal{S}^{\alpha,i}$ is the quasi-periodic single layer potential with the wave number $k = i$. In fact, let $v \in H^1(Y \setminus \partial D)$ be the quasi-periodic single layer potential given by

$$v(x) = \int_{\partial D} G^{\alpha,k}(x, y) \phi(y) d\sigma(y), \quad \phi \in H^{-\frac{1}{2}}(\partial D), \quad x \in Y \setminus \partial D.$$

Clearly, v satisfies Helmholtz equation in D and $Y \setminus \overline{D}$ and the α -quasi-periodic boundary condition on ∂Y

$$\begin{aligned} v(1, x_2) &= e^{i\alpha_1} v(0, x_2), & x_2 \in (0, 1), \\ v(x_1, 1) &= e^{i\alpha_2} v(x_1, 0), & x_1 \in (0, 1), \end{aligned}$$

$$\begin{aligned}\frac{\partial v}{\partial x_1}(1, x_2) &= e^{i\alpha_1} \frac{\partial v}{\partial x_1}(0, x_2), \quad x_2 \in (0, 1), \\ \frac{\partial v}{\partial x_2}(x_1, 1) &= e^{i\alpha_2} \frac{\partial v}{\partial x_2}(x_1, 0), \quad x_1 \in (0, 1).\end{aligned}$$

Setting $k = i$ in the definition of v , and applying Green's first identity, we find that

$$\begin{aligned}(\mathcal{S}^{\alpha, i}[\phi], \phi)_{\frac{1}{2}, -\frac{1}{2}} &= \int_{\partial D} \left(\frac{\partial v}{\partial \nu} \Big|_+ - \frac{\partial v}{\partial \nu} \Big|_- \right) \bar{v} d\sigma \\ &= \int_{Y \setminus \bar{D}} (|\nabla v|^2 + |v|^2) \bar{v} dx - \int_{\partial Y} \bar{v} \frac{\partial v}{\partial \nu} d\sigma + \int_D (|\nabla v|^2 + |v|^2) \bar{v} dx.\end{aligned}$$

Since v is quasi-periodic function, we can see $\int_{\partial Y} \bar{v} \frac{\partial v}{\partial \nu} d\sigma = 0$ (see [25] page 125), and then

$$(\mathcal{S}^{\alpha, i}[\phi], \phi)_{\frac{1}{2}, -\frac{1}{2}} = \int_{Y \setminus \bar{D}} (|\nabla v|^2 + |v|^2) \bar{v} dx + \int_D (|\nabla v|^2 + |v|^2) \bar{v} dx. \quad (6.1)$$

Furthermore, from the jump relations of v and trace theorem, it follows that

$$\|\phi\|_{H^{-\frac{1}{2}}(\partial D)} = \left\| \frac{\partial v}{\partial \nu} \Big|_+ - \frac{\partial v}{\partial \nu} \Big|_- \right\|_{H^{-\frac{1}{2}}(\partial D)} \lesssim \|v\|_{H^1(D)} + \|v\|_{H^1(Y \setminus \bar{D})}. \quad (6.2)$$

Thus, combining (6.1) and (6.2), it yields

$$\|\phi\|_{H^{-\frac{1}{2}}(\partial D)} \lesssim (\mathcal{S}^{\alpha, i}[\phi], \phi)_{\frac{1}{2}, -\frac{1}{2}}.$$

Notice that the integral kernel of $\mathcal{S}^{\alpha, k} - \mathcal{S}^{\alpha, i}$ is a C^∞ function in a neighborhood of $\partial D \times \partial D$ and hence we conclude that $\mathcal{S}^{\alpha, k} - \mathcal{S}^{\alpha, i}$ is compact from $H^{-\frac{1}{2}}(\partial D)$ to $H^{\frac{1}{2}}(\partial D)$. Applying the Lax-Milgram lemma to the bounded and coercive sesquilinear form

$$a(\phi, \psi) := (\mathcal{S}^{\alpha, i}[\phi], \psi)_{\frac{1}{2}, -\frac{1}{2}}, \quad \phi, \psi \in H^{-\frac{1}{2}}(\partial D).$$

we imply that $(\mathcal{S}^{\alpha, i})^{-1} : H^{\frac{1}{2}}(\partial D) \rightarrow H^{-\frac{1}{2}}(\partial D)$ exists and is bounded. Using the Theorem 5.14 in [26] and compactness of $\mathcal{S}^{\alpha, k} - \mathcal{S}^{\alpha, i}$, it finds that $\mathcal{S}^{\alpha, k}$ is an isomorphism if and only if $\mathcal{S}^{\alpha, k}$ is injective. However, the injectivity of $\mathcal{S}^{\alpha, k}$ is proved in Lemma 3.3. Thus, the statements in Lemma 3.4 are proved. \square

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